

Topological Quantum Field Theory and Seiberg-Witten Monopoles

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Abstract

A topological quantum field theory is introduced which reproduces the Seiberg-Witten invariants of four-manifolds. Dimensional reduction of this topological field theory leads to a new one in three dimensions. Its partition function yields a three-manifold invariant, which can be regarded as the Seiberg-Witten version of Casson's invariant. A Geometrical interpretation of the three dimensional quantum field theory is also given.

1 INTRODUCTION

The field of low dimensional geometry and topology[2] has undergone a dramatic phase of progress in recent years, prompted, to a large extent, by new ideas and discoveries in mathematical physics. The discovery of quantum groups[7] in the study of the Yang-Baxter equation[4] has reshaped the theory of knots and links[10][14][22]; the study of conformal field theory and quantum Chern-Simons theory[18] in physics had a profound impact on the theory of three-manifolds; and most importantly, investigations of the classical Yang-Mills theory led to the creation of the Donaldson theory of four-manifolds[9][6]. Very recently, Witten[19] discovered a new set of invariants of four-manifolds in the study of the Seiberg-Witten monopole equations, which have their origin in supersymmetric gauge theory. The Seiberg-Witten theory, while closely related to Donaldson theory, is much easier to handle. Using Seiberg-Witten theory, proofs of many theorems in Donaldson theory have been simplified, and several important new results have also been obtained[15] [16][11]. However, being only a several

months old, Seiberg-Witten theory remains to be fully developed. Solutions to any of the outstanding problems, e.g., the precise relationship to Donaldson theory, will undoubtedly have great impact on the theory of four-manifolds.

As is well known, Donaldson theory can be cast into the framework of topological quantum field theory[20][5]. This formulation provides a useful tool for exploring conceptual aspects of the theory, and in some cases, e.g., for Kahler manifolds, it even enables explicit computations of the Donaldson invariants.

Our aim here is to study the Seiberg-Witten invariants and related three-manifold invariants using topological quantum field theory techniques. We first construct a four dimensional topological quantum field theory which reproduces the Seiberg-Witten invariants as correlation functions, and also examine some of its properties. Then we dimensionally reduce it to obtain a three-dimensional topological quantum field theory, the correlation functions of which give rise to invariants of three-manifolds. The partition function of the three-dimensional theory is study in some detail; in particular, we cast it into the Atiyah-Jeffrey framework, thus to provide it with an interpretation in pure geometrical terms.¹

2 TOPOLOGICAL FIELD THEORY

2.1 Seiberg-Witten invariants and monopole equations

The Seiberg-Witten monopole equations are classical field theoretical equations involving a $U(1)$ gauge field and a complex Weyl spinor on a four dimensional manifold. Let X denote the four-manifold, which is assumed to be oriented and closed. If X is spin, there exist positive and negative spin bundles S^\pm of rank two. Introduce a complex line bundle $L \rightarrow X$. Let A be a connection on L and M be a section of the product bundle $S^+ \otimes L$. The Seiberg-Witten monopole equations read

$$\begin{aligned} F_{kl}^+ &= -\frac{i}{2} \bar{M} \Gamma_{kl} M, \\ D_A M &= 0, \end{aligned} \tag{1}$$

where D_A is the twisted Dirac operator, $\Gamma_{ij} = \frac{1}{2}[\gamma_i, \gamma_j]$, and F^+ represents the self-dual part of the curvature of L with connection A .

If X is not a spin manifold, then spin bundles do not exist. However, it is always possible to introduce the so called $Spin_c$ bundles $S^\pm \otimes L$, with L^2 being a line bundle. Then in this more general setting, the Seiberg-Witten monopoles equations look formally the same as (1), but the M should be interpreted as a section of the the $Spin_c$ bundle $S^+ \otimes L$.

Denote by \mathcal{M} the moduli space of solutions of the Seiberg-Witten monopole equations up to gauge transformations. Generically, this space is a manifold. Its virtual dimension is equal to the number of solutions of the following equations

$$\begin{aligned} (d\psi)_{kl}^+ + \frac{i}{2} (\bar{M} \Gamma_{kl} N + \bar{N} \Gamma_{kl} M) &= 0, \\ D_A N + \psi M &= 0, \end{aligned}$$

¹While this paper was being finished we noticed that Labastida and Mariño [12] also obtained the similar result in 4-dimensional case.

$$\nabla_k \psi^k + \frac{i}{2}(\overline{N}M - \overline{M}N) = 0, \quad (2)$$

where A and M are a given solution of (1), $\psi \in \Omega^1(X)$ is a one form, $(d\psi)^+ \in \Omega^{2,+}(X)$ is the self dual part of the two form $d\psi$, and $N \in S^+ \otimes L$. The first two of the equations in (2) are the linearization of the monopole equations (1), while the last one is a gauge fixing condition. Though with a rather unusual form, it arises naturally from the dual operator governing gauge transformations

$$\begin{aligned} C : \Omega^0(X) &\rightarrow \Omega^1(X) \oplus (S^+ \otimes L) \\ \phi &\mapsto (-d\phi, i\phi M). \end{aligned}$$

Let

$$T : \Omega^1(X) \oplus (S^+ \otimes L) \rightarrow \Omega^0(X) \oplus \Omega^{2,+}(X) \oplus (S^- \otimes L), \quad (3)$$

be the operator governing equation (2), namely, the operator which allows us to rewrite (2) as

$$T(\psi, N) = 0.$$

Then T is an elliptic operator, the index $Ind(T)$ of which yields the virtual dimension of \mathcal{M} . A straightforward application of the Atiyah-Singer index theorem gives

$$Ind(T) = -\frac{2\chi(X) + 3\sigma(X)}{4} + c_1(L)^2,$$

where $\chi(X)$ is the Euler character of X , $\sigma(X)$ its signature index and $c_1(L)^2$ is the square of the first Chern class of L evaluated on X in the standard way.

When $Ind(T)$ equals zero, the moduli space generically consists of a finite number of points, $\mathcal{M} = \{ \sqrt{\sqcup} : \sqcup = \infty, \in, \dots, \mathcal{I} \}$. Let ϵ_t denote the sign of the determinant of the operator T at p_t , which can be defined with mathematical rigour. Then the Seiberg-Witten invariant of the four-manifold X is defined by

$$\sum_1^I \epsilon_t. \quad (4)$$

The fact that this is indeed an invariant(i.e., independent of the metric) of X is not very difficult to prove, and we refer to [19] for details. As a matter of fact, the number of solutions of a system of equations weighted by the sign of the operator governing the equations(i.e., the analog of T) is a topological invariant in general[19]. This point of view has been extensively explored by Vafa and Witten[21] within the framework of topological quantum field theory in connection with the so called S duality. Here we wish to explore the Seiberg-Witten invariants following a similar line as that taken in [20][21].

2.2 Topological Lagrangian

Introduce a Lie superalgebra with an odd generator Q and two even generators U and δ obeying the following (anti)commutation relations

$$[U, Q] = Q, \quad [Q, Q] = 2\delta, \quad [Q, \delta] = 0. \quad (5)$$

We will call U the ghost number operator, and Q the BRST operator.

Let A be a connection of L and $M \in S^+ \otimes L$. We define the action of the superalgebra on these fields by requiring that δ coincide with a gauge transformation with a gauge parameter $\phi \in \Omega^0(X)$. The field multiplets associated with A and M furnishing representations of the superalgebra are (A, ψ, ϕ) , and (M, N) , where $\psi \in \Omega^1(X)$, $\phi \in \Omega^0(X)$, and N is a section of $S^+ \otimes L$. They transform under the action of the superalgebra according to

$$\begin{aligned} [Q, A_i] &= \psi_i, & [Q, M] &= N, \\ [Q, \psi_i] &= -\partial_i \phi, & [Q, N] &= i\phi M, \\ [Q, \phi] &= 0. \end{aligned} \tag{6}$$

We assume that both A and M have ghost number 0, and thus will be regarded as bosonic fields when we study their quantum field theory. The ghost numbers of other fields can be read off the above transformation rules. We have that ψ and N are of ghost number 1, thus are fermionic, and ϕ is of ghost number 2 and bosonic. Note that the multiplet (A, ψ, ϕ) is what one would get in the topological field theory for Donaldson invariants except that our gauge group is $U(1)$, while the existence of M and N is a new feature. Also note that both M and ψ have the wrong statistics.

In order to construct a quantum field theory which will reproduce the Seiberg-Witten invariants as correlation functions, anti-ghosts and Lagrangian multipliers are also required. We introduce the anti-ghost multiplet $(\lambda, \eta) \in \Omega^0(X)$, such that

$$[U, \lambda] = -2\lambda, \quad [Q, \lambda] = \eta, \quad [Q, \eta] = 0, \tag{7}$$

and the Lagrangian multipliers $(\chi, H) \in \Omega^{2,+}(X)$, and $(\mu, \nu) \in S^- \otimes L$ such that

$$\begin{aligned} [U, \chi] &= -\chi, & [Q, \chi] &= H, & [Q, H] &= 0; \\ [U, \mu] &= -\mu, & [Q, \mu] &= \nu, & [Q, \nu] &= i\phi\mu. \end{aligned} \tag{8}$$

With the given fields, we construct the following functional which has ghost number -1 :

$$\begin{aligned} V &= \int_X \left\{ [\nabla_k \psi^k + \frac{i}{2}(\overline{N}M - \overline{M}N)]\lambda - \chi^{kl} \left(H_{kl} - F_{kl}^+ - \frac{i}{2}\overline{M}\Gamma_{kl}M \right) \right. \\ &\quad \left. - \bar{\mu}(\nu - iD_A M) - \overline{(\nu - iD_A M)}\mu \right\}, \end{aligned} \tag{9}$$

where the indices of the tensorial fields are raised and lowered by a given metric g on X , and the integration measure is the standard $\sqrt{g}d^4x$. Also, \overline{M} and $\bar{\mu}$ etc. represent the hermitian conjugate of the spinorial fields. In a formal language, $\overline{M} \in S^+ \otimes L^{-1}$ and $\bar{\mu}, \bar{\nu}, \overline{D_A M} \in S^- \otimes L^{-1}$. Following the standard procedure in constructing topological quantum field theory, we take the classical action of our theory to be

$$S = [Q, V], \tag{10}$$

which has ghost number 0. One can easily show that S is also BRST invariant, i.e.,

$$[Q, S] = 0, \tag{11}$$

thus it is invariant under the full superalgebra(5).

The bosonic Lagrangian multiplier fields H and ν do not have any dynamics, and so can be eliminated from the action by using their equations of motion

$$\begin{aligned} H_{kl} &= \frac{1}{2} \left(F_{kl}^+ + \frac{i}{2} \bar{M} \Gamma_{kl} M \right), \\ \nu &= \frac{1}{2} i D_A M. \end{aligned} \quad (12)$$

Then we arrive at the following expression for the action

$$\begin{aligned} S &= \int_X \left\{ [-\Delta\phi + \bar{M}M\phi - i\bar{N}N]\lambda - [\nabla_k\psi^k + \frac{i}{2}(\bar{N}M - \bar{M}N)]\eta + 2i\phi\bar{\mu}\mu \right. \\ &+ \overline{(iD_A N - \gamma.\psi M)}\mu - \bar{\mu}(iD_A N - \gamma.\psi M) \\ &- \chi^{kl} \left[(\nabla_k\psi^l - \nabla_l\psi^k)^+ + \frac{i}{2}(\bar{M}\Gamma_{kl}N + \bar{N}\Gamma_{kl}M) \right] \Big\} \\ &+ S_0, \end{aligned} \quad (13)$$

where S_0 is given by

$$S_0 = \int_X \left\{ \frac{1}{4} |F^+|^2 + \frac{i}{2} \bar{M} \Gamma M|^2 + \frac{1}{2} |D_A M|^2 \right\}. \quad (14)$$

It is interesting to observe that S_0 is nonnegative, and vanishes if and only if A and M satisfy the Seiberg-Witten monopole equations. As pointed out in [19], S_0 can be rewritten as

$$S_0 = \int_X \left\{ \frac{1}{4} |F^+|^2 + \frac{1}{4} |M|^4 + \frac{1}{8} R |M|^2 + g^{ij} \bar{D}_i \bar{M} D_j M \right\},$$

where R is the scalar curvature of X associated with the metric g . If R is nonnegative over the entire X , then the only square integrable solution of the monopole equations (1) is A is a anti-self-dual connection and $M = 0$.

2.3 Quantum theory

We will now investigate the quantum field theory defined by the classical action (13) with the path integral method. Let \mathcal{F} collectively denote all the fields. The partition function of the theory is defined by

$$Z = \int \mathcal{D}\mathcal{F} \exp(-\frac{\infty}{\epsilon} S),$$

where $e \in \mathbf{R}$ is the coupling constant. The integration measure $\mathcal{D}\mathcal{F}$ is defined on the space of all the fields. However, since S is invariant under the gauge transformations, we assume the integration over the gauge field to be performed over the gauge orbits of A . In other words, we fix a gauge for the A field using, say, a Faddeev-Popov type procedure. This can be carried out in the standard manner, thus there is no need for us to spell out the details here. The integration measure $\mathcal{D}\mathcal{F}$ can be shown to be invariant under the super charge Q . Also, it does not explicitly involve the metric g of X .

Let W be any operator in the theory. Its correlation function is defined by

$$Z[W] = \int \mathcal{D}\mathcal{F} \exp(-\frac{\infty}{\hbar\epsilon} \mathcal{S}) \mathcal{W}.$$

It follows from the Q invariance of both the action S and the path integration measure that for any operator W ,

$$\begin{aligned} Z[[Q, W]] &= \int \mathcal{D}\mathcal{F} \exp(-\frac{\infty}{\hbar\epsilon} \mathcal{S}) [Q, \mathcal{W}] \\ &= 0. \end{aligned}$$

For the purpose of constructing topological invariants of the four- manifold X , we are particularly interested in operators W which are BRST closed,

$$[Q, W] = 0, \tag{15}$$

but not BRST exact, i.e., can not be expressed as the (anti)- commutators of Q with other operators. For such a W , if its variation with respect to the metric g is *BRST* exact,

$$\delta_g W = [Q, W'], \tag{16}$$

then its correlation function $Z[W]$ is a topological invariant of X (by that we really mean that it does not depend on the metric g):

$$\begin{aligned} \delta_g Z[W] &= \int \mathcal{D}\mathcal{F} \exp(-\frac{\infty}{\hbar\epsilon} \mathcal{S}) [Q, \mathcal{W}' - \frac{\infty}{\hbar\epsilon} \delta_{\mathcal{J}} \mathcal{V} \cdot \mathcal{W}] \\ &= 0. \end{aligned}$$

In particular, the partition function Z itself is a topological invariant. In fact, under certain conditions imposed on X , the partition function coincides with the Seiberg-Witten invariants, as we will prove below.

Another important property of the partition function is that it does not depend on the coupling constant e :

$$\begin{aligned} \frac{\partial Z}{\partial e^2} &= \int \mathcal{D}\mathcal{F} \frac{\infty}{\hbar\Delta} \exp(-\frac{\infty}{\hbar\epsilon} \mathcal{S}) [Q, \mathcal{V}] \\ &= 0. \end{aligned}$$

Therefore, Z can be computed *exactly* in the limit when the coupling constant goes to zero. Such a computation can be carried out in the standard way: Let A^o , M^o be a solution of the equations of motion of A and M arising from the action S . We expand the fields A and M around this classical configuration,

$$A = A^o + ea, \quad M = M^o + em,$$

where a and m are the quantum fluctuations of A and M respectively. All the other fields do not acquire background components, thus are purely quantum mechanical.

We scale them by the coupling constant e , by setting N to eN , ϕ to $e\phi$ etc.. To the order $o(1)$ in e^2 , we have

$$Z = \sum_p \exp\left(-\frac{1}{e^2} S_{cl}^{(p)}\right) \int \mathcal{DF}' \exp(-\mathcal{S}_{\Pi}^{(\vee)}), \quad (17)$$

where $S_q^{(p)}$ is the quadratic part of the action in the quantum fields and depends on the gauge orbit of the classical configuration A^o , M^o , which we label by p . Explicitly,

$$\begin{aligned} S_q^{(p)} = & \int_X \left\{ [-\Delta\phi + \bar{M}^o M^o \phi - i\bar{N}N]\lambda - [\nabla_k \psi^k + \frac{i}{2}(\bar{N}M^o - \bar{M}^o N)]\eta + 2i\phi\bar{\mu}\mu \right. \\ & + \frac{(iD_{A^o}N - \gamma.\psi M^o)\mu - \bar{\mu}(iD_{A^o}N - \gamma.\psi M^o)}{2} \\ & - \chi^{kl} \left[(\nabla_k \psi^l - \nabla_l \psi^k)^+ + \frac{i}{2}(\bar{M}^o \Gamma_{kl}N + \bar{N}\Gamma_{kl}M^o) \right] \\ & \left. + \frac{1}{4}|f^+ + \frac{i}{2}(\bar{m}\Gamma M^o + \bar{M}^o \Gamma m)|^2 + \frac{1}{2}|iD_{A^o}m + \gamma.aM^o|^2 \right\}, \end{aligned}$$

with f^+ the self-dual part of $f = da$. The classical part of the action is given by

$$S_{cl}^{(p)} = S_0|_{A=A^o, M=M^o}.$$

The integration measure \mathcal{DF}' has exactly the same form as \mathcal{DF} but with A replaced by a , and M by m , \bar{M} by \bar{m} respectively. Needless to say, the summation over p runs through all gauge classes of classical configurations.

Let us now examine further features of our quantum field theory. A gauge class of classical configurations may give a non-zero contribution to the partition function in the limit $e^2 \rightarrow 0$ only if $S_{cl}^{(p)}$ vanishes, and this happens if and only if A^o and M^o satisfy (1). Therefore, the Seiberg-Witten monopole equations are recovered from the quantum field theory.

The equations of motion of the fields ψ and N in the semi-classical approximation can be easily derived from the quadratic action $S_q^{(p)}$, solutions of which are the zero modes of the quantum fields ψ and N . The equations of motion read

$$\begin{aligned} (d\psi)_{kl}^+ + \frac{i}{2}(\bar{M}^o \Gamma_{kl}N + \bar{N}\Gamma_{kl}M^o) &= 0, \\ D_{A^o}N + \gamma.\psi M^o &= 0, \\ \nabla_k \psi^k + \frac{i}{2}(\bar{N}M^o - \bar{M}^o N) &= 0. \end{aligned} \quad (18)$$

Note that they are exactly the same equations which we have already discussed in (2). The first two equations are the linearization of the monopole equations, while the last is a ‘gauge fixing condition’ for ψ . The dimension of the space of solutions of these equations is the virtual dimension of the moduli space \mathcal{M} . Thus, within the context of our quantum field theoretical model, the virtual dimension of \mathcal{M} is identified with the number of the zero modes of the quantum fields ψ and N .

For simplicity we assume that there are no zero modes of ψ and N , i.e., the moduli space is zero dimensional. Then no zero modes exist for the other two fermionic fields χ and μ . To compute the partition function in this case, we first observe that the quadratic action $S_q^{(p)}$ is invariant under the supersymmetry obtained by expanding Q

to first order in the quantum fields around the monopole solution A^o , M^o (equations of motion for the nonpropagating fields H and ν should also be used.). This supersymmetry transforms the set of 8 real bosonic fields (each complex field is counted as two real ones; the a_i contribute 2 upon gauge fixing.) and the set of 16 fermionic fields to each other. Thus at a given monopole background we obtain

$$\int \mathcal{D}\mathcal{F}' \exp(-\mathcal{S}_{\Pi}^{(\vee)}) = \frac{\text{Pfaff}(\nabla_F)}{|\text{Pfaff}(\nabla_F)|} = \epsilon^{(p)},$$

where $\epsilon^{(p)}$ is $+1$ or -1 . In the above equation, ∇_F is the skew symmetric first order differential operator defining the fermionic part of the action $S_q^{(p)}$, which can be read off from $S_q^{(p)}$ to be $\nabla_F = \begin{pmatrix} 0 & T \\ -T^* & 0 \end{pmatrix}$. Therefore, $\epsilon^{(p)}$ is the sign of the determinant of the elliptic operator T at the monopole background A^o , M^o , and the partition function

$$Z = \sum_p \epsilon^{(p)}, \quad (19)$$

coincides with the Seiberg-Witten invariant of the four-manifold X .

When the dimension of the moduli space \mathcal{M} is greater than zero, the partition function Z vanishes identically, due to integration over zero modes of the fermionic fields. In order to obtain any non trivial topological invariants for the underlying manifold X , we need to examine correlations functions of operators satisfying equations (15) and (16). A class of such operators can be constructed following the standard procedure[19]. We define the following set of operators

$$\begin{aligned} W_{k,0} &= \frac{\phi^k}{k!}, \\ W_{k,1} &= \psi W_{k-1,0}, \\ W_{k,2} &= F W_{k-1,0} - \frac{1}{2} \psi \wedge \psi W_{k-2,0}, \\ W_{k,3} &= F \wedge \psi W_{k-2,0} - \frac{1}{3!} \psi \wedge \psi \wedge \psi W_{k-3,0}, \\ W_{k,4} &= \frac{1}{2} F \wedge F W_{k-2,0} - \frac{1}{2} F \wedge \psi \wedge \psi W_{k-3,0} - \frac{1}{4!} \psi \wedge \psi \wedge \psi \wedge \psi W_{k-4,0}. \end{aligned} \quad (20)$$

These operators are clearly independent of the metric g of X . Although they are not BRST invariant except for $W_{k,0}$, they obey the following equations

$$\begin{aligned} dW_{k,0} &= -[Q, W_{k,1}], \\ dW_{k,1} &= [Q, W_{k,2}], \\ dW_{k,2} &= -[Q, W_{k,3}], \\ dW_{k,3} &= [Q, W_{k,4}], \\ dW_{k,4} &= 0, \end{aligned}$$

which allow us to construct BRST invariant operators from the the W 's in the following way: Let X_i , $i = 1, 2, 3$, $X_4 = X$, be compact manifolds without boundary embedded in X . We assume that these submanifolds are homologically nontrivial. Define

$$\begin{aligned} \hat{O}_{k,0} &= W_{k,0}, \\ \hat{O}_{k,i} &= \int_{X_i} W_{k,i}, \quad i = 1, 2, 3, 4. \end{aligned} \quad (21)$$

As we have already pointed out, $\widehat{O}_{k,0}$ is BRST invariant. It follows from the descendent equations that

$$\begin{aligned} [Q, \widehat{O}_{k,i}] &= \int_{X_i} [Q, W_{k,i}] \\ &= \int_{X_i} dW_{k,i-1} = 0. \end{aligned}$$

Therefore the operators \widehat{O} indeed have the properties (15) and (16). Also, for the boundary ∂K of an $i + 1$ dimensional manifold K embedded in X , we have

$$\begin{aligned} \int_{\partial K} W_{k,i} &= \int_K dW_{k,i} \\ &= [Q, \int_K W_{k,i+1}], \end{aligned}$$

is BRST trivial. The correlation function of $\int_{\partial K} W_{k,i}$ with any BRST invariant operator is identically zero. This in particular shows that the \widehat{O} 's only depend on the homological classes of the submanifolds X_i .

3 DIMENSIONAL REDUCTION

In this section we dimensionally reduce the quantum field theoretical model for the Seiberg-Witten invariant from four dimensions to three dimensions, thus to obtain a new topological quantum field theory defined on 3- manifolds. Its partition function yields a 3- manifold invariant, which can be regarded as the Seiberg-Witten version of Casson's invariant [1][17].

3.1 Three-dimensional field theory

We take the four-manifold X to be of the form $Y \times [0, 1]$ with Y being a compact 3-manifold without boundary. The metric on X will be taken to be

$$(ds)^2 = (dt)^2 + \sum_{i,j}^3 g_{ij}(x) dx^i dx^j,$$

where the 'time' t -independent $g(x)$ is the Riemannian metric on Y . We assume that Y admits a spin structure which is compatible with the $Spin_c$ structure of X , i.e., if we think of Y as embedded in X , then this embedding induces maps from the $Spin_c$ bundles $S^\pm \otimes L$ of X to $\tilde{S} \otimes L$, where \tilde{S} is a spin bundle and L is a line bundle over Y .

To perform the dimensional reduction, we impose the condition that all fields are t in dependent. This leads to the following action

$$\begin{aligned} S &= \int \sqrt{g} d^3x \left\{ [-\Delta\phi + \overline{M}M\phi - i\overline{N}N]\lambda - [\nabla_k\psi^k + \frac{i}{2}(\overline{N}M - \overline{M}N)]\eta + 2i\phi\bar{\mu}\mu \right. \\ &+ \overline{[i(D_A + b)N - (\sigma.\psi - \tau)M]}\mu - \bar{\mu}[i(D_A + b)N - (\sigma.\psi - \tau)M] \\ &- 2\chi^k [-\partial_k\tau + *(\nabla\psi)_k - \bar{M}\sigma_k N - \bar{N}\sigma_k M] \\ &\left. + \frac{1}{4}|*F - \partial b - \bar{M}\sigma M|^2 + \frac{1}{2}|(D_A + b)M|^2 \right\}, \end{aligned} \tag{22}$$

where the k is a three-dimensional index, and σ_k are the Pauli matrices. The fields $b, \tau \in \Omega^0(Y)$ respectively arose from A_0 and ψ_0 of the four dimensional theory, while the meanings of the other fields are clear. The BRST symmetry in four-dimensions carries over to the three-dimensional theory. The BRST transformations rules for (A_i, ψ_i, ϕ) , $i = 1, 2, 3$, (M, N) , and (λ, η) are the same as before, but for the other fields, we have

$$\begin{aligned} [Q, b] &= \tau, \\ [Q, \tau] &= 0, \\ [Q, \chi_k] &= \frac{1}{2} \left(*F_k - \partial_k b - \bar{M} \sigma_k M \right), \\ [Q, \mu] &= \frac{1}{2} i (D_A + b) M. \end{aligned} \tag{23}$$

The action S is cohomological in the sense that $S = [Q, V_3]$, with V_3 being the dimensionally reduced version of V defined by (9), and $[Q, S] = 0$. Thus it gives rise to a topological field theory upon quantization. The partition function of the theory

$$Z = \int \mathcal{D}\mathcal{F} \exp(-\frac{\infty}{\epsilon} S),$$

can be computed exactly in the limit $e^2 \rightarrow 0$, as it is coupling constant independent. We have, as before,

$$Z = \sum_p \exp(-\frac{1}{e^2} S_{cl}^{(p)}) \int \mathcal{D}\mathcal{F}' \exp(-\mathcal{S}_{\Pi}^{(\vee)}),$$

where $S_q^{(p)}$ is the quadratic part of S expanded around a classical configuration with the classical parts for the fields A, M, b being A^o, M^o, b^o , while those for all the other fields being zero. The classical action $S_{cl}^{(p)}$ is given by

$$S_{cl}^{(p)} = \int_Y \left\{ \frac{1}{4} |*F^o - db^o - \bar{M}^o \sigma M^o|^2 + \frac{1}{2} |(D_{A^o} + b^o) M^o|^2 \right\},$$

which can be rewritten as

$$S_{cl}^{(p)} = \int_Y \left\{ \frac{1}{4} |*F^o - \bar{M}^o \sigma M^o|^2 + \frac{1}{2} |D_{A^o} M^o|^2 + \frac{1}{2} |db^o|^2 + \frac{1}{2} |b^o M^o|^2 \right\}.$$

In order for the classical configuration to have nonvanishing contributions to the partition function, all the terms in $S_{cl}^{(p)}$ should vanish separately. Therefore,

$$\begin{aligned} *F^o - \bar{M}^o \sigma M^o &= 0, \\ D_{A^o} M^o &= 0, \end{aligned} \tag{24}$$

and

$$b^o = 0,$$

where the last condition requires some explanation. When we have a trivial solution of the equations (24), it can be replaced by the less stringent condition $db^o = 0$.

However, in a more rigorous treatment of the problem at hand, we in general perturb the equations (24), then the trivial solution does not arise.

Let us define an operator

$$\begin{aligned} \tilde{T} : \Omega^0(Y) \oplus \Omega^1(Y) \oplus (\tilde{S} \otimes L) &\rightarrow \Omega^0(Y) \oplus \Omega^1(Y) \oplus (\tilde{S} \otimes L), \\ (\tau, \psi, N) &\mapsto (-d^*\psi + \frac{i}{2}(\bar{N}M - \overline{M}N), \quad *(d\psi) - d\tau - \bar{N}\sigma M - \overline{M}\sigma N, \\ &\quad iD_A N - (\sigma.\psi - \tau)M), \end{aligned} \quad (25)$$

where the complex bundle $\tilde{S} \otimes L$ should be regarded as a real one with twice the rank. This operator is self-adjoint, and is also obviously elliptic. We will assume that it is Fredholm as well. In terms of \tilde{T} , the equations of motion of the fields χ^i and μ can be expressed as

$$\tilde{T}^{(p)}(\tau, \psi, N) = 0, \quad (26)$$

where $\tilde{T}^{(p)}$ is the operator \tilde{T} with the background fields (A^o, M^o) belonging to the gauge class p of classical configurations.

When the kernel of \tilde{T} is zero, the partition function Z does not vanish identically. An easy computation leads to

$$Z = \sum_p \epsilon^{(p)},$$

where the sum is over all gauge inequivalent solutions of (24), and $\epsilon^{(p)}$ is the sign of the determinant of $\tilde{T}^{(p)}$.

A rigorous definition of the sign of the $\det(\tilde{T})$ can be devised. However, if we are to compute only the absolute value of Z , then it is sufficient to know the sign of $\det(\tilde{T})$ relative to a fixed gauge class of classical configurations. This can be achieved using the $\text{mod} - 2$ spectral flow of a family of Fredholm operators \tilde{T}_t along a path of solutions of (24). More explicitly, let (A^o, M^o) belong to the gauge class of classical configurations p , and $(\tilde{A}^o, \tilde{M}^o)$ in \tilde{p} . We consider the solution of the Seiberg-Witten equation on $X = Y \times [0, 1]$ with $A_0 = 0$ and also satisfying the following conditions

$$\begin{aligned} (A, M)|_{t=0} &= (A^o, M^o), \\ (A, M)|_{t=1} &= (\tilde{A}^o, \tilde{M}^o). \end{aligned}$$

Using this solution in \tilde{T} results in a family of Fredholm operators, which has zero kernels at $t = 0$ and 1 . The spectral flow of \tilde{T}_t , denoted by $q(p, \tilde{p})$, is defined to be the number of eigenvalues which cross zero with a positive slope minus the number which cross zero with a negative slope. This number is a well defined quantity, and is given by the index of the operator $\frac{\partial}{\partial t} - \tilde{T}_t$. In terms of the spectral flow, we have

$$\frac{\det(\tilde{T}^{(p)})}{\det(\tilde{T}^{(\tilde{p})})} = (-1)^{q(p, \tilde{p})}.$$

Equations (24) can be derived from the functional

$$S_{c-s} = \frac{1}{2} \int_Y A \wedge F + i \int_Y \sqrt{g} d^3x \overline{M} D_A M.$$

(It is interesting to observe that this is almost the standard Lagrangian of a $U(1)$ Chern-Simons theory coupled to spinors, except that we have taken M to have bosonic statistics.) S_{c-s} is gauge invariant modulus a constant arising from the Chern-Simons term upon a gauge transformation. Therefore, $(\frac{\delta S_{c-s}}{\delta A}, \frac{\delta S_{c-s}}{\delta M})$ defines a vector field on the quotient space of all $U(1)$ connections \mathcal{A} tensored with the $\tilde{S} \times L$ sections by the $U(1)$ gauge group \mathcal{G} , i.e., $\mathcal{W} = (\mathcal{A} \times (\tilde{\mathcal{S}} \otimes \mathcal{L})) / \mathcal{G}$. Solutions of (24) are zeros of this vector field, and $\tilde{T}^{(p)}$ is the Hessian at the point $p \in \mathcal{W}$. Thus the partition Z is nothing else but the Euler character of \mathcal{W} . This geometrical interpretation will be spelt out more explicitly in the next subsection by re-interpreting the theory using the Mathai-Quillen formula[13].

3.2 Geometrical interpretation

To elucidate the geometric meaning of the three-dimensional theory obtained in the last section, we now cast it into the framework of Atiyah and Jeffrey[3]. Let us briefly recall the geometric set up of the Mathai-Quillen formula as reformulated in reference [3]. Let P be a Riemannian manifold of dimension $2m + \dim G$, and G be a compact Lie group acting on P by isometries. Then $P \rightarrow P/G$ is a principle bundle. Let V be a $2m$ dimensional real vector space, which furnishes a representation $G \rightarrow SO(2m)$. Form the associated vector bundle $P \times_G V$. Now the Thom form of $P \times_G V$ can be expressed

$$\begin{aligned} U &= \frac{\exp(-x^2)}{(2\pi)^{\dim G} \pi^m} \int \exp \left\{ \frac{i\chi\phi\chi}{4} + i\chi dx - i\langle \delta\nu, \lambda \rangle \right. \\ &\quad \left. - \langle \phi, R\lambda \rangle + \langle \nu, \eta \rangle \right\} \mathcal{D}\eta \mathcal{D}\chi \mathcal{D}\phi \mathcal{D}\lambda, \end{aligned} \quad (27)$$

where $x = (x^1, \dots, x^{2m})$ is the coordinates of V , ϕ and λ are bosonic variables in the Lie algebra \mathfrak{g} of G , and η and χ are Grassmannian variables valued in the Lie algebra and the tangent space of the fiber respectively. In the above equation, C maps any $\eta \in \mathfrak{g}$ to the element of the vertical part of TP generated by η ; ν is the \mathfrak{g} -valued one form on P defined by $\langle \nu(\alpha), \eta \rangle = \langle \alpha, C(\eta) \rangle$, for all vector fields α ; and $R = C^*C$. Also, δ is the exterior derivative on P .

Now we choose a G invariant map $s : P \rightarrow V$, and pull back the Thom form U . Then the top form on P in s^*U is the Euler class. If $\{\delta p\}$ forms a basis of the cotangent space of P (note that ν and δs are one forms on P), we replace it by a set of Grassmannian variables $\{\psi\}$ in s^*U , then integrate them away. We arrive at

$$\begin{aligned} &\frac{1}{(2\pi)^{\dim G} \pi^m} \int \exp \left\{ -|s|^2 + \frac{i\chi\phi\chi}{4} + i\chi\delta s - i\langle \delta\nu, \lambda \rangle \right. \\ &\quad \left. - \langle \phi, R\lambda \rangle + \langle \psi, C\eta \rangle \right\} \mathcal{D}\eta \mathcal{D}\chi \mathcal{D}\phi \mathcal{D}\lambda \mathcal{D}\psi, \end{aligned} \quad (28)$$

the precise relationship of which with the Euler character of $P \times_G V$ is

$$\int_P (28) = \text{Vol}(G) \chi(P \times_G V).$$

It is rather obvious that the action S defined by (13) for the four-dimensional theory can be interpreted as the exponent in the integrand of (28), if we identify P

with $\mathcal{A} \times -(\mathcal{W}^+)$, and V with $\Omega^{2,+}(X) \times \Gamma(W^-)$, and set $s = (F^+ + \frac{i}{2}\bar{M}\Gamma M, D_A M)$. Here \mathcal{A} is the space of all $U(1)$ connections of $\det(W^+)$, and $\Gamma(W^\pm)$ are the sections of $S^\pm \otimes L$ respectively.

For the three-dimensional theory, we wish to show that the partition function yields the Euler number of \mathcal{W} . However, the tangent bundle of \mathcal{W} cannot be regarded as an associated bundle with the principal bundle, for which for the formulae (27) or (28) can readily apply, some further work is required.

Let P be the principal bundle over P/G , V, V' be two orthogonal representations of G . Suppose there is an embedding from $P \times_G V'$ to $P \times_G V$ via a G -map $\gamma(p) : V' \rightarrow V$ for $p \in P$. Denote the resulting quotient bundle as E . In order to derive the Thom class for E , one needs to choose a section of E , or equivalently, a G -map $s : P \rightarrow V$ such that $s(p) \in (\text{Im } \gamma(p))^\perp$. Then the Euler class of E can be expressed as $\pi_* \rho^* U$, where U is the Thom class of $P \times_G V$, ρ is a G -map: $P \times V' \rightarrow P \times V$ defined by

$$\rho(p, \tau) = (p, \gamma(p)\tau + s(p)),$$

and π_* is the integration along the fiber for the projection $\pi : P \times V' \rightarrow P/G$. Explicitly,

$$\begin{aligned} \pi_* \rho^*(U) &= \int \exp \left\{ -|\gamma(p)\tau + s(p)|^2 + i\chi\phi\chi + i\chi\delta(\gamma(p)\tau + s(p)) \right. \\ &\quad \left. -i\langle \delta\nu, \lambda \rangle - \langle \phi, R\lambda \rangle + \langle \nu, C\eta \rangle \right\} \mathcal{D}\chi \mathcal{D}\phi \mathcal{D}\tau \mathcal{D}\eta \mathcal{D}\lambda \end{aligned} \quad (29)$$

Consider the exact sequence

$$0 \longrightarrow (\mathcal{A} \times -(\mathcal{W})) \times_{\mathcal{G}} \otimes'(\mathcal{Y}) \xrightarrow{\quad} (\mathcal{A} \times -(\mathcal{W})) \times_{\mathcal{G}} (\otimes^\infty(\mathcal{Y}) \times -(\mathcal{W}))$$

where $j_{(\mathcal{A}, M)} : b \mapsto (-db, bM)$. (We assume that $M \neq 0$). Then the tangent bundle of $\mathcal{A} \times_{\mathcal{G}} -(\mathcal{W})$ can be Regarded as the quotient bundle

$$(\mathcal{A} \times -(\mathcal{W})) \times_{\mathcal{G}} (\otimes^\infty(\mathcal{Y}) \times -(\mathcal{W})) / \mathcal{I} \Downarrow (|).$$

We define a vector field on $\mathcal{A} \times_{\mathcal{G}} -(\mathcal{W})$ by

$$s(A, M) = (*F_A - \bar{M}\sigma M, D_A M),$$

which lies in $(\text{Im } j)^\perp$:

$$\int_Y (*F_A - \bar{M}\sigma M) \wedge *(-db) + \int_Y \sqrt{g} d^3x \langle D_A M, bM \rangle = 0, \quad (30)$$

where we have used the short hand notation $\langle M_1, M_2 \rangle = \frac{1}{2}(\bar{M}_1 M_2 + \bar{M}_2 M_1)$.

Formally applying the formula (29) to the present infinite dimensional situation, we obtain the Euler calss $\pi_* \rho^*(U)$ for the tangent bundle $T(\mathcal{A} \times_{\mathcal{G}} -(\mathcal{W}))$, where ρ is the \mathcal{G} -invariant map ρ is defined by

$$\begin{aligned} \rho : \quad \Omega^0(Y) &\longrightarrow \Omega^1(Y) \times \Gamma(W), \\ \rho(b) &= (-db + *F_A - \bar{M}\sigma M, (D_A + b)M), \end{aligned}$$

π is the projection $(\mathcal{A} \times -(\mathcal{W})) \times_{\mathcal{G}} \otimes'(\mathcal{Y}) \longrightarrow \mathcal{A} \times_{\mathcal{G}} -(\mathcal{W})$, and π_* signifies the integration along the fiber. Also U is the Thom form of the bundle

$$(\mathcal{A} \times -(\mathcal{W})) \times_{\mathcal{G}} (\otimes^\infty(\mathcal{Y}) \times -(\mathcal{W})) \longrightarrow \mathcal{A} \times_{\mathcal{G}} -(\mathcal{W}).$$

To get a concrete feel about U , we need to explain the geometry of this bundle. The metric on Y and the hermitian metric $\langle \cdot, \cdot \rangle$ on $\Gamma(W)$ naturally define a connection. The Maurer-Cartan connection on $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}$ is flat while the hermitian connection on \mathcal{W} has the curvature $i\phi\mu \wedge \bar{\mu}$. This gives the expression of term $i(\chi, \mu)\phi(\chi, \mu)$ in (28) in our case.

In our infinite dimensional setting, the map C is given by

$$\begin{aligned} C : \quad \Omega^0(Y) &\longrightarrow T_{(A,M)}(\mathcal{A} \times -(\mathcal{W})) \\ C(\eta) &= (-d\eta, i\eta M), \end{aligned}$$

and its dual is given by

$$\begin{aligned} C^* : \quad \Omega^1(Y) \times \Gamma(W) &\longrightarrow \Omega^0(Y), \\ C^*(\psi, N) &= -d^*\psi + \langle N, iM \rangle. \end{aligned}$$

The one form $\langle \nu, \eta \rangle$ on $\mathcal{A} \times -(\mathcal{W})$ takes the value

$$\langle (\psi, N), C\eta \rangle = \langle -d^*\psi, \eta \rangle + \langle N, iM \rangle \eta$$

on the vector field (ψ, N) . We also easily obtain $R(\lambda) = -\Delta\lambda + \langle M, M \rangle \lambda$, where $\Delta = d^*d$. The $\langle \delta\nu, \lambda \rangle$ is a two form on $\mathcal{A} \times -(\mathcal{W})$ whose value on $(\psi_1, N_1), (\psi_2, N_2)$ is $-\langle N_1, N_2 \rangle \lambda$.

Combining all the information together, we arrive at the following formula,

$$\begin{aligned} \pi_*\rho^*(U) &= \int \exp \left\{ -\frac{1}{2}|\rho|^2 + i(\chi, \mu)\delta\rho + 2i\phi\mu\bar{\mu} \right. \\ &\quad + \langle \Delta\phi, \lambda \rangle - \phi\lambda\langle M, M \rangle + i\langle N, N \rangle \lambda \\ &\quad \left. + \langle \nu, \eta \rangle \right\} \mathcal{D}\chi \mathcal{D}\phi \mathcal{D}\lambda \mathcal{D}\eta. \end{aligned} \quad (31)$$

Note that the 1-form $i(\chi, \mu)\delta\rho$ on $\mathcal{A} \times -(\mathcal{W}) \times \otimes'(\mathcal{Y})$ contacted with the vector field (ϕ, N, b) leads to

$$\begin{aligned} &2\chi^k \left[-\partial_k \tau + *(\nabla\psi)_k - \bar{M}\sigma_k N - \bar{N}\sigma_k M \right] + \\ &2\langle \mu, [i(D_A + b)N - (\sigma\psi - \tau)M] \rangle; \end{aligned} \quad (32)$$

and the relation (30) gives $|\rho|^2 = |*F - \bar{M}\sigma M|^2 + |db|^2 + |D_A M|^2 + b^2|M|^2$. Finally we obtain the Euler character

$$\pi_*\rho^*(U) = \int \exp(-S) \mathcal{D}\chi \mathcal{D}\phi \mathcal{D}\lambda \mathcal{D}\eta \mathcal{D} \quad (33)$$

where S is the action (22) of the three dimensional theory defined on the manifold Y .

Integrating (33) over $\mathcal{A} \times_{\mathcal{G}} -(\mathcal{W})$ leads to Euler number

$$\sum_{[(A,M)]:s(A,M)=0} \epsilon^{(A,M)}.$$

which coincides with the partition function Z of our three-dimensional theory(recall that Z is independent of the coupling constant.).

References

- [1] S. Akbulut and J. D. McCarthy, *Casson's invariant for homological 3-spheres-An exposition*, Mathematical Notes **36**, Princeton Univ. Press, Princeton(1990).
- [2] M. F. Atiyah, *New invariants of three and four dimensional manifolds*, in The Mathematical Heritage of Hermann Weyl, eds. R. Well et al., Proc. Symp. Pure. Math. **48** (Am. Math. Soc., Providence, 1988).
- [3] M. Atiyah and L. Jeffrey, Topological Lagrangians and Cohomology, *J. Geom. Phys.***7** (1990) 119.
- [4] R. J. Baxter, *Exactly solved models in statistical mechanics* (Academic Press, 1982).
- [5] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Topological field theory*, Phys. Reports **209** (1991) 129-340.
- [6] S. K. Donaldson and P. B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, Clarendon, Oxford(1990).
- [7] V.G. Drinfeld, *Quantum groups*, Proc. ICM, Berkeley, (1986)798; M. Jimbo, Lett. Math. Phys. **10**(1985)63.
- [8] A. Floer, An instanton invariant for three-manifolds, Commun. Math. Phys. **118** (1988) 215.
- [9] D. Freed and K. K. Uhlenbeck, *Instantons and four manifolds*, Springer-Verlag, New York(1984).
- [10] V. F. R. Jones, Bull. Amer. Math. Soc. **12** (1985) 103.
- [11] P. B. Kronheimer and T. S. Mrowka, *The genus of embedded surfaces in projective plane*, Math. Res. Lett. **1** (1995) 797-808.
- [12] Labastida J. M. F., and Mariño M., *A Topological Lagrangian For Monopole on Four Manifolds* preprint hep-th/9503105.
- [13] Mathai V. and D. Quillen, *Super connections, Thom classes and equivariant differential forms*, Topology **25** (1986) 85-110.
- [14] N. Yu Reshetikhin and V. G. Turaev, Invent. Math. **10** (1991) 547.
- [15] C. H. Taubes, *The Seiberg-Witten and Gromov invariants*, Preprint(1995).
- [16] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. **1** (1994) 809-822.
- [17] C. H. Taubes, *Casson's invariant and gauge theory*, J. Diff. Geom. **31** (1990) 547-599.
- [18] E. Witten, Commun. Math. Phys. **121** (1989) 351.

- [19] E. Witten, Monopoles and four manifolds, *Math. Res. Lett.***1**(1994) 769.
- [20] E. Witten, Topological quantum field theory, *Commun. Math. Phys.* **117** (1988) 353.
- [21] C. Vafa and E. Witten, A strong coupling test of S duality, preprint, hep-th/9408074.
- [22] R.B. Zhang, M.D. Gould and A.J. Bracken, *Commun. Math. Phys.*, **137** (1991) 13.